Discrete Mathematics

mathematical structures

tutorialspoint

SIMPLYEASYLEARNING

www.tutorialspoint.com

https://www.facebook.com/tutorialspointindia

https://twitter.com/tutorialspoint

About the Tutorial

Discrete Mathematics is a branch of mathematics involving discrete elements that uses algebra and arithmetic. It is increasingly being applied in the practical fields of mathematics and computer science. It is a very good tool for improving reasoning and problem-solving capabilities.

This tutorial explains the fundamental concepts of Sets, Relations and Functions, Mathematical Logic, Group theory, Counting Theory, Probability, Mathematical Induction and Recurrence Relations, Graph Theory, Trees and Boolean Algebra.

Audience

This tutorial has been prepared for students pursuing a degree in any field of computer science and mathematics. It endeavors to help students grasp the essential concepts of discrete mathematics.

Prerequisites

This tutorial has an ample amount of both theory and mathematics. The readers are expected to have a reasonably good understanding of elementary algebra and arithmetic.

Copyright & Disclaimer

© Copyright 2016 by Tutorials Point (I) Pvt. Ltd.

All the content and graphics published in this e-book are the property of Tutorials Point (I) Pvt. Ltd. The user of this e-book is prohibited to reuse, retain, copy, distribute or republish any contents or a part of contents of this e-book in any manner without written consent of the publisher.

We strive to update the contents of our website and tutorials as timely and as precisely as possible, however, the contents may contain inaccuracies or errors. Tutorials Point (I) Pvt. Ltd. provides no guarantee regarding the accuracy, timeliness or completeness of our website or its contents including this tutorial. If you discover any errors on our website or in this tutorial, please notify us at contact@tutorialspoint.com

tutorialspoint
SIMPLYEASYLEARNING

Table of Contents

About the Tutorial i
Audience i
Prerequisites i
Copyright & Disclaimer i
Table of Contents ii

1. Discrete Mathematics – Introduction 1

PART 1: SETS, RELATIONS, AND FUNCTIONS 2

2. Sets 3
Set – Definition 3
Representation of a Set 3
Cardinality of a Set 4
Types of Sets 5
Venn Diagrams 7
Set Operations 7
Power Set 9
Partitioning of a Set 9

3. Relations 11
Definition and Properties 11
Domain and Range 11
Representation of Relations using Graph 11
Types of Relations 12

4. Functions 14
Function – Definition 14
Injective / One-to-one function 14
Surjective / Onto function 14
Bijective / One-to-one Correspondent 14
Inverse of a Function 15
Composition of Functions 15

PART 2: MATHEMATICAL LOGIC 17

5. Propositional Logic 18
Prepositional Logic – Definition 18
Connectives 18
Tautologies 20
Contradictions 20
Contingency 20
Propositional Equivalences 21
Inverse, Converse, and Contra-positive 22
Duality Principle 22
Normal Forms 22

tutorialspoint
SIMPLYEASYLEARNING

6. Predicate Logic 24
Predicate Logic – Definition 24
Well Formed Formula 24
Quantifiers 24
Nested Quantifiers 25

7. Rules of Inference 26
What are Rules of Inference for? 26
Table of Rules of Inference 26
Addition 27
Conjunction 27
Simplification 27
Modus Ponens 27
Modus Tollens 28
Disjunctive Syllogism 28
Hypothetical Syllogism 28
Constructive Dilemma 29
Destructive Dilemma 29

PART 3: GROUP THEORY 30

8. Operators and Postulates 31
Closure 31
Associative Laws 31
Commutative Laws 32
Distributive Laws 32
Identity Element 32
Inverse 33
De Morgan's Law 33

9. Group Theory 34
Semigroup 34
Monoid 34
Group 34
Abelian Group 35
Cyclic Group and Subgroup 35
Partially Ordered Set (POSET) 36
Hasse Diagram 37
Linearly Ordered Set 37
Lattice 38
Properties of Lattices 39
Dual of a Lattice 39

PART 4: COUNTING & PROBABILITY 40

10. Counting Theory 41
The Rules of Sum and Product 41
Permutations 41
Combinations 43
Pascal's Identity 44
Pigeonhole Principle 44
The Inclusion-Exclusion principle 45

tutorialspoint
SIMPLYEASYLEARNING

11. Probability 46
Basic Concepts 46
Probability Axioms 47
Properties of Probability 48
Conditional Probability 48
Bayes' Theorem 49

PART 5: MATHEMATICAL INDUCTION & RECURRENCE RELATIONS 51

12. Mathematical Induction 52
Definition 52
How to Do It 52
Strong Induction 54

13. Recurrence Relation 55
Definition 55
Linear Recurrence Relations 55
Non-Homogeneous Recurrence Relation and Particular Solutions 57
Generating Functions 59

PART 6: DISCRETE STRUCTURES 60

14. Graph and Graph Models 61
What is a Graph? 61
Types of Graphs 62
Representation of Graphs 66
Planar vs. Non-planar Graph 68
Isomorphism 68
Homomorphism 69
Euler Graphs 69
Hamiltonian Graphs 70

15. More on Graphs 71
Graph Coloring 71
Graph Traversal 72

16. Introduction to Trees 77
Tree and its Properties 77
Centers and Bi-Centers of a Tree 77
Labeled Trees 80
Unlabeled trees 80
Rooted Tree 81
Binary Search Tree 82

17. Spanning Trees 84
Minimum Spanning Tree 85
Kruskal's Algorithm 85
Prim's Algorithm 89

tutorialspoint
SIMPLYEASYLEARNING

PART 7: BOOLEAN ALGEBRA 92

18. Boolean Expressions and Functions 93
Boolean Functions 93
Boolean Expressions 93
Boolean Identities 93
Canonical Forms 94
Logic Gates 96

19. Simplification of Boolean Functions 99
Simplification Using Algebraic Functions 99
Karnaugh Maps 100
Simplification Using K- map 101

tutorialspoint
SIMPLYEASYLEARNING

1. Discrete Mathematics – Introduction

Mathematics can be broadly classified into two categories:

- **Continuous Mathematics** – It is based upon continuous number line or the real numbers. It is characterized by the fact that between any two numbers, there are almost always an infinite set of numbers. For example, a function in continuous mathematics can be plotted in a smooth curve without breaks.

- **Discrete Mathematics** – It involves distinct values; i.e. between any two points, there are a countable number of points. For example, if we have a finite set of objects, the function can be defined as a list of ordered pairs having these objects, and can be presented as a complete list of those pairs.

Topics in Discrete Mathematics

Though there cannot be a definite number of branches of Discrete Mathematics, the following topics are almost always covered in any study regarding this matter:

- Sets, Relations and Functions
- Mathematical Logic
- Group theory
- Counting Theory
- Probability
- Mathematical Induction and Recurrence Relations
- Graph Theory
- Trees
- Boolean Algebra

We will discuss each of these concepts in the subsequent chapters of this tutorial.

Part 1: Sets, Relations, and Functions

tutorialspoint
SIMPLYEASYLEARNING

2. Sets

German mathematician **G. Cantor** introduced the concept of sets. He had defined a set as a collection of definite and distinguishable objects selected by the means of certain rules or description.

Set theory forms the basis of several other fields of study like counting theory, relations, graph theory and finite state machines. In this chapter, we will cover the different aspects of **Set Theory**.

Set – Definition

A set is an unordered collection of different elements. A set can be written explicitly by listing its elements using set bracket. If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.

Some Example of Sets

- A set of all positive integers
- A set of all the planets in the solar system
- A set of all the states in India
- A set of all the lowercase letters of the alphabet

Representation of a Set

Sets can be represented in two ways:

- Roster or Tabular Form
- Set Builder Notation

Roster or Tabular Form

The set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.

Example 1: Set of vowels in English alphabet, A = {a,e,i,o,u}

Example 2: Set of odd numbers less than 10, B = {1,3,5,7,9}

tutorialspoint
SIMPLYEASYLEARNING

Set Builder Notation

The set is defined by specifying a property that elements of the set have in common. The set is described as A = { x : p(x)}

Example 1: The set {a,e,i,o,u} is written as:
A = { x : x is a vowel in English alphabet}

Example 2: The set {1,3,5,7,9} is written as:
B = { x : 1≤x<10 and (x%2) ≠ 0}

If an element x is a member of any set S, it is denoted by x∈ S and if an element y is not a member of set S, it is denoted by y ∉ S.

Example: If S = {1, 1.2,1.7,2}, 1∈ S but 1.5 ∉S

Some Important Sets

N: the set of all natural numbers = {1, 2, 3, 4, ...}

Z: the set of all integers = {....., -3, -2, -1, 0, 1, 2, 3, ...}

Z^+: the set of all positive integers

Q: the set of all rational numbers

R: the set of all real numbers

W: the set of all whole numbers

Cardinality of a Set

Cardinality of a set S, denoted by |S|, is the number of elements of the set. The number is also referred as the cardinal number. If a set has an infinite number of elements, its cardinality is ∞.

Example: |{1, 4, 3,5}| = 4, |{1, 2, 3,4,5,…}| = ∞

If there are two sets X and Y,

- |X| = |Y| denotes two sets X and Y having same cardinality. It occurs when the number of elements in X is exactly equal to the number of elements in Y. In this case, there exists a bijective function 'f' from X to Y.

- |X| ≤ |Y| denotes that set X's cardinality is less than or equal to set Y's cardinality. It occurs when number of elements in X is less than or equal to that of Y. Here, there exists an injective function 'f' from X to Y.

tutorialspoint
SIMPLYEASYLEARNING

- |X| < |Y| denotes that set X's cardinality is less than set Y's cardinality. It occurs when number of elements in X is less than that of Y. Here, the function 'f' from X to Y is injective function but not bijective.

- If |X| ≤ |Y| and |Y| ≤ |X| then |X| = |Y|. The sets X and Y are commonly referred as equivalent sets.

Types of Sets

Sets can be classified into many types. Some of which are finite, infinite, subset, universal, proper, singleton set, etc.

Finite Set

A set which contains a definite number of elements is called a finite set.

Example: S = {x | x ∈ N and 70 > x > 50}

Infinite Set

A set which contains infinite number of elements is called an infinite set.

Example: S = {x | x ∈ N and x > 10}

Subset

A set X is a subset of set Y (Written as X ⊆ Y) if every element of X is an element of set Y.

Example 1: Let, X = { 1, 2, 3, 4, 5, 6 } and Y = { 1, 2 }. Here set Y is a subset of set X as all the elements of set Y is in set X. Hence, we can write Y ⊆ X.

Example 2: Let, X = {1, 2, 3} and Y = {1, 2, 3}. Here set Y is a subset (Not a proper subset) of set X as all the elements of set Y is in set X. Hence, we can write Y ⊆ X.

Proper Subset

The term "proper subset" can be defined as "subset of but not equal to". A Set X is a proper subset of set Y (Written as X ⊂ Y) if every element of X is an element of set Y and |X| < |Y|.

Example: Let, X = {1, 2, 3, 4, 5, 6} and Y = {1, 2}. Here set Y ⊂ X since all elements in Y are contained in X too and X has at least one element is more than set Y.

tutorialspoint
SIMPLYEASYLEARNING

Universal Set

It is a collection of all elements in a particular context or application. All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U.

Example: We may define U as the set of all animals on earth. In this case, set of all mammals is a subset of U, set of all fishes is a subset of U, set of all insects is a subset of U, and so on.

Empty Set or Null Set

An empty set contains no elements. It is denoted by ∅. As the number of elements in an empty set is finite, empty set is a finite set. The cardinality of empty set or null set is zero.

Example: $S = \{x \mid x \in N \text{ and } 7 < x < 8\} = \emptyset$

Singleton Set or Unit Set

Singleton set or unit set contains only one element. A singleton set is denoted by {s}.

Example: $S = \{x \mid x \in N, 7 < x < 9\} = \{ 8 \}$

Equal Set

If two sets contain the same elements they are said to be equal.

Example: If A = {1, 2, 6} and B = {6, 1, 2}, they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

Equivalent Set

If the cardinalities of two sets are same, they are called equivalent sets.

Example: If A = {1, 2, 6} and B = {16, 17, 22}, they are equivalent as cardinality of A is equal to the cardinality of B. i.e. |A|=|B|=3

Overlapping Set

Two sets that have at least one common element are called overlapping sets.

In case of overlapping sets:

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$

tutorialspoint
SIMPLYEASYLEARNING

- $n(A) = n(A - B) + n(A \cap B)$
- $n(B) = n(B - A) + n(A \cap B)$

Example: Let, A = {1, 2, 6} and B = {6, 12, 42}. There is a common element '6', hence these sets are overlapping sets.

Disjoint Set

Two sets A and B are called disjoint sets if they do not have even one element in common. Therefore, disjoint sets have the following properties:

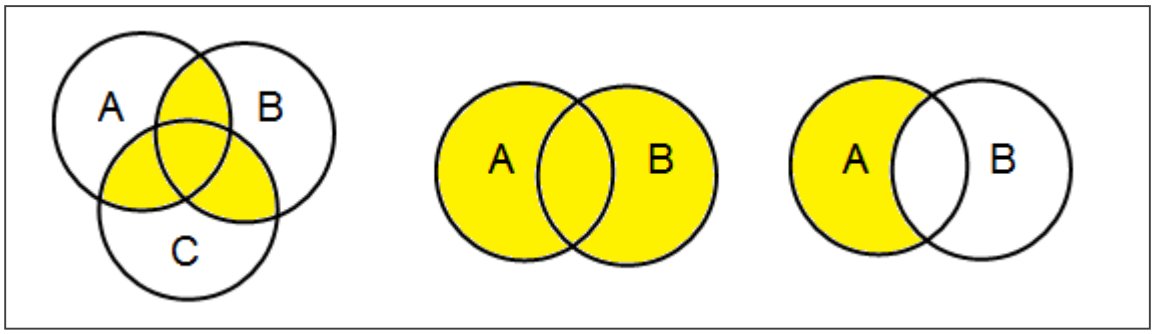
- $n(A \cap B) = \emptyset$
- $n(A \cup B) = n(A) + n(B)$

Example: Let, A = {1, 2, 6} and B = {7, 9, 14}; there is not a single common element, hence these sets are overlapping sets.

Venn Diagrams

Venn diagram, invented in1880 by John Venn, is a schematic diagram that shows all possible logical relations between different mathematical sets.

Examples



Set Operations

Set Operations include Set Union, Set Intersection, Set Difference, Complement of Set, and Cartesian Product.

Set Union

The union of sets A and B (denoted by A ∪ B) is the set of elements which are in A, in B, or in both A and B. Hence, A∪B = {x | x ∈A OR x ∈B}.

tutorialspoint
SIMPLYEASYLEARNING

Example: If A = {10, 11, 12, 13} and B = {13, 14, 15}, then A ∪ B = {10, 11, 12, 13, 14, 15}. (The common element occurs only once)

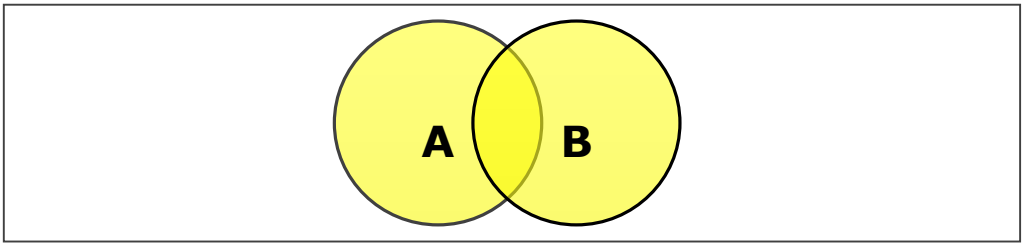


Figure: Venn Diagram of A ∪ B

Set Intersection

The intersection of sets A and B (denoted by A ∩ B) is the set of elements which are in both A and B. Hence, A∩B = {x | x ∈A AND x ∈B}.

Example: If A = {11, 12, 13} and B = {13, 14, 15}, then A∩B = {13}.

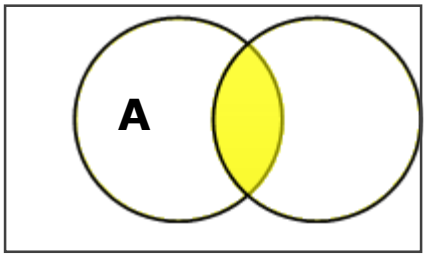
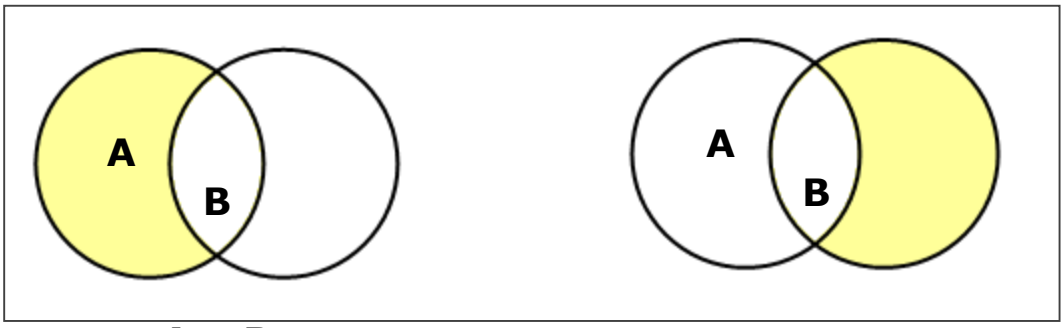


Figure: Venn Diagram of A ∩ B

Set Difference/ Relative Complement

The set difference of sets A and B (denoted by A–B) is the set of elements which are only in A but not in B. Hence, A−B = {x | x ∈A AND x ∉B}.

Example: If A = {10, 11, 12, 13} and B = {13, 14, 15}, then (A−B) = {10, 11, 12} and (B−A) = {14,15}. Here, we can see (A−B) ≠ (B−A)



tutorialspoint
SIMPLYEASYLEARNING

Figure: Venn Diagram of A – B and B – A

Complement of a Set

The complement of a set A (denoted by A') is the set of elements which are not in set A. Hence, A' = {x | x ∉A}.

More specifically, A'= (U–A) where U is a universal set which contains all objects.

Example: If A ={x | x belongs to set of odd integers} then A' ={y | y does not belong to set of odd integers}

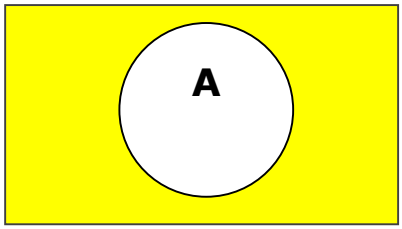


Figure: Venn Diagram of A'

Cartesian Product / Cross Product

The Cartesian product of n number of sets A_1, A_2.....A_n, denoted as $A_1 \times A_2 \times ..... \times A_n$, can be defined as all possible ordered pairs (x_1,x_2,....x_n) where $x_1 \in A_1$, $x_2 \in A_2$, …… $x_n \in A_n$

Example: If we take two sets A= {a, b} and B= {1, 2},

The Cartesian product of A and B is written as: A×B= {(a, 1), (a, 2), (b, 1), (b, 2)}

The Cartesian product of B and A is written as: B×A= {(1, a), (1, b), (2, a), (2, b)}

Power Set

Power set of a set S is the set of all subsets of S including the empty set. The cardinality of a power set of a set S of cardinality n is 2^n. Power set is denoted as P(S).

Example:

For a set S = {a, b, c, d} let us calculate the subsets:

- Subsets with 0 elements: {∅} (the empty set)
- Subsets with 1 element: {a}, {b}, {c}, {d}
- Subsets with 2 elements: {a,b}, {a,c}, {a,d}, {b,c}, {b,d},{c,d}
- Subsets with 3 elements: {a,b,c},{a,b,d},{a,c,d},{b,c,d}
- Subsets with 4 elements: {a,b,c,d}

Hence, P(S) =

tutorialspoint
SIMPLYEASYLEARNING

{ {∅},{a}, {b}, {c}, {d},{a,b}, {a,c}, {a,d}, {b,c},
{b,d},{c,d},{a,b,c},{a,b,d},{a,c,d},{b,c,d},{a,b,c,d} **}**

| P(S) | = 2^4 =16

Note: The power set of an empty set is also an empty set.
| P ({∅}) | = 2^0 = 1

Partitioning of a Set

Partition of a set, say *S*, is a collection of *n* disjoint subsets, say $P_1, P_2, \ldots\ldots P_n$, that satisfies the following three conditions:

- P_i does not contain the empty set.
  $[P_i \neq \{\emptyset\}$ for all $0 < i \leq n]$
- The union of the subsets must equal the entire original set.
  $[P_1 \cup P_2 \cup \ldots\ldots \cup P_n = S]$
- The intersection of any two distinct sets is empty.
  $[P_a \cap P_b = \{\emptyset\}$, for $a \neq b$ where $n \geq a, b \geq 0$]

Example

Let S = {a, b, c, d, e, f, g, h}

One probable partitioning is {a}, {b, c, d}, {e, f, g,h}

Another probable partitioning is {a,b}, { c, d}, {e, f, g,h}

Bell Numbers

Bell numbers give the count of the number of ways to partition a set. They are denoted by B_n where n is the cardinality of the set.

Example:

Let S = { 1, 2, 3}, n = |S| = 3

The alternate partitions are:

1. ∅, {1, 2, 3}
2. {1}, {2, 3}
3. {1, 2}, {3}
4. {1, 3}, {2}
5. {1}, {2},{3}

tutorialspoint
SIMPLYEASYLEARNING

Hence $B_3 = 5$

tutorialspoint
SIMPLYEASYLEARNING

3. Relations

Whenever sets are being discussed, the relationship between the elements of the sets is the next thing that comes up. **Relations** may exist between objects of the same set or between objects of two or more sets.

Definition and Properties

A binary relation R from set x to y (written as xRy or R(x,y)) is a subset of the Cartesian product x × y. If the ordered pair of G is reversed, the relation also changes.

Generally an n-ary relation R between sets A_1, ... , and A_n is a subset of the n-ary product $A_1 \times ... \times A_n$. The minimum cardinality of a relation R is Zero and maximum is n^2 in this case.

A binary relation R on a single set A is a subset of A × A.

For two distinct sets, A and B, having cardinalities *m* and *n* respectively, the maximum cardinality of a relation R from A to B is *mn*.

Domain and Range

If there are two sets A and B, and relation R have order pair (x, y), then:

- The **domain** of R, Dom(R), is the set { x | (x, y) ∈ R for some y in B }
- The **range** of R, Ran(R), is the set { y | (x, y) ∈ R for some x in A }

Examples

Let, A = {1,2,9} and B = {1,3,7}

- Case 1: If relation R is 'equal to' then R = {(1, 1), (3, 3)}
  Dom(R) = { 1, 3}, Ran(R) = { 1, 3}
- Case 2: If relation R is 'less than' then R = {(1, 3), (1, 7), (2, 3), (2, 7)}
  Dom(R) = { 1, 2}, Ran(R) = { 3, 7}
- Case 3: If relation R is 'greater than' then R = {(2, 1), (9, 1), (9, 3), (9, 7)}
  Dom(R) = { 2, 9}, Ran(R) = { 1, 3, 7}

tutorialspoint
SIMPLYEASYLEARNING

Representation of Relations using Graph

A relation can be represented using a directed graph.

The number of vertices in the graph is equal to the number of elements in the set from which the relation has been defined. For each ordered pair (x, y) in the relation R, there will be a directed edge from the vertex 'x' to vertex 'y'. If there is an ordered pair (x, x), there will be self- loop on vertex 'x'.

Suppose, there is a relation R = {(1, 1), (1,2), (3, 2)} on set S = {1,2,3}, it can be represented by the following graph:

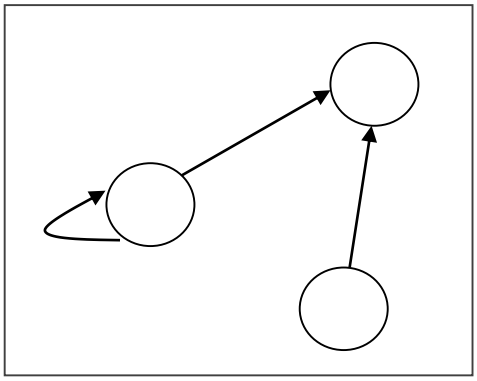


Figure: Representation of relation by directed graph

Types of Relations

1. The **Empty Relation** between sets X and Y, or on E, is the empty set ∅
2. The **Full Relation** between sets X and Y is the set X×Y
3. The **Identity Relation** on set X is the set {(x,x) | x ∈ X}
4. The Inverse Relation R' of a relation R is defined as: R'= {(b,a) | (a,b) ∈R}

   Example: If R = {(1, 2), (2,3)} then R' will be {(2,1), (3,2)}

5. A relation R on set A is called **Reflexive** if ∀a∈A is related to a (aRa holds).

   Example: The relation R = {(a,a), (b,b)} on set X={a,b} is reflexive

6. A relation R on set A is called **Irreflexive** if no a∈A is related to a (aRa does not hold).

   Example: The relation R = {(a,b), (b,a)} on set X={a,b} is irreflexive

7. A relation R on set A is called **Symmetric** if xRy implies yRx, ∀x∈A and ∀y∈A.

   Example: The relation R = {(1, 2), (2, 1), (3, 2), (2, 3)} on set A={1, 2, 3} is symmetric.

8. A relation R on set A is called **Anti-Symmetric** if xRy and yRx implies

tutorialspoint
SIMPLYEASYLEARNING

$x=y \quad \forall x \in A$ and $\forall y \in A$.

Example: The relation $R = \{ (x,y) \in N \mid x \leq y \}$ is anti-symmetric since $x \leq y$ and $y \leq x$ implies $x = y$.

9. A relation R on set A is called **Transitive** if xRy and yRz implies xRz, $\forall x,y,z \in A$.

   Example: The relation R = {(1, 2), (2, 3), (1, 3)} on set A= {1, 2, 3} is transitive.

10. A relation is an **Equivalence Relation** if it is reflexive, symmetric, and transitive.

    Example: The relation R = {(1, 1), (2, 2), (3, 3), (1, 2),(2,1), (2,3), (3,2), (1,3), (3,1)} on set A= {1, 2, 3} is an equivalence relation since it is reflexive, symmetric, and transitive.

tutorialspoint
SIMPLYEASYLEARNING

End of ebook preview
If you liked what you saw…
Buy it from our store @ **https://store.tutorialspoint**

tutorialspoint
SIMPLYEASYLEARNING